

# Comparison of Closed-form Solutions for the Lucas-Uzawa model via the Partial Hamiltonian Approach and the Classical Approach

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## Abstract

In this paper we derive the closed-form solutions for the Lucas-Uzawa growth model with the aid of the partial Hamiltonian approach and then compare our results with those derived by the classical approach [1]. The partial Hamiltonian approach provides two first integrals [2] in the case where there are no parameter restrictions and these two first integrals are utilized to construct three sets of closed form solutions for all the variables in the model. First two first integrals are used to find two closed form solutions, one of which is new to the literature. We then use only one of the first integrals to derive a third solution that is the same as that found in the previous literature. We also show that all three solutions converge to the same long run balance growth path.

**Keywords:** Economic growth, Partial or current value Hamiltonian approach, Lucas-Uzawa model, Current-value Hamiltonian

## 1 Introduction

One of the foundations of modern economic growth theory is the two-sector endogenous growth model developed by Lucas and Uzawa. The model addresses the relationship between human capital accumulation and economic growth and the idea behind the Lucas-Uzawa model ([3] and [4]) is to determine optimal time paths for consumption and the amount of labor devoted to

the production of capital in an economy which has constrained levels of physical and human capital. One of the interesting features of the Lucas-Uzawa model is the existence of multiple equilibria.

The partial Hamiltonian approach [5] uses tools from Lie group theory and is used to construct closed-form solutions of dynamical systems such as those arising in economic growth theory. This approach is unique and a significant departure from the rest of the literature because unlike the previously used methods, the partial Hamiltonian is applicable to an arbitrary system of ordinary differential equations which means that it can be applied to more complex models [6]. In the context of our paper, the partial Hamiltonian methodology yields a series of first integrals for a system of ordinary differential equations and we use these first integrals to find closed-form solutions for the Lucas-Uzawa model. In this paper we establish the closed-form solutions for the Lucas-Uzawa model with the aid of the partial Hamiltonian approach and we compare our results with those derived by the classical approach [1]. The partial Hamiltonian approach provides two first integrals [2] for case where there are no parameter restrictions. We utilize these two first integrals to construct closed form solutions for all variables of model for two different scenarios: (i)  $z = z^*$  and (ii)  $z \neq z^*$  where  $z(t) = \frac{h(t)u(t)}{k(t)}$ .

We begin by using both first integrals to construct closed-form solutions for both scenarios. One solution is exactly the same as derived by Chilarescu [1] and the second is completely new to the literature. We then use only one first integral to determine a different solution, again under fairly general parameter values, and show that this is the same solution that has been derived in the previous literature (see Chilarescu [1]).

We find that in the case where  $z = z^*$ , both the partial Hamiltonian approach and the classical approach provide one solution. For the  $z \neq z^*$  case, the classical approach yields one solution while the partial Hamiltonian approach yields the same solution as well as providing one additional solution which is completely new to the literature. What is especially interesting about our new solution is that while the equilibrium levels of consumption and capital stock in the new solution are equal to those found in the old solution, the amount of labor allocated to the production of physical capital and the level of human capital are different in the new solution. The existence of three closed form solutions is new to the literature and we also show that these closed form solutions all converge to the same long run balanced growth path.

It is important to mention here that under a specific parameter restriction

$\sigma = \frac{\beta(\rho+\pi)}{2\pi\beta-\delta+\delta\beta-\pi}$  a third first integral was obtained and the closed form solution for this case was new in the literature (see Naz et al [2]). The partial Hamiltonian approach provides three solutions for the case in which there are no parameter restrictions. For the case in which the specific parameter restriction  $\sigma = \frac{\beta(\rho+\pi)}{2\pi\beta-\delta+\delta\beta-\pi}$  is imposed, there is an additional solution as shown in Naz et al [5]. The classical approach provides only two closed-form solutions for this model (see Chilarescu [1]) which means that the partial Hamiltonian approach not only provides all the solutions constructed in the previous literature (see Chilarescu [1]) but also provides additional closed-form solutions which are completely new to the literature.

The layout of the paper is as follows. In Section 2, we introduce the Lucas-Uzawa model and provided expressions for the two first integrals derived previously in [5]. In Section 3, the closed-form solutions of the dynamical system of ODEs are constructed by utilizing both first integrals in the case of no parameter restrictions. In Section 4, we use only one first integral to derive the closed form solutions for all the variables in the model. A comparison of our results with those derived by the classical approach is presented in Section 5. Finally, our conclusions are summarized in Section 6.

## 2 The Lucas-Uzawa model

The representative agent's utility function is defined as

$$Max_{c,u} \int_0^\infty \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t}, \sigma \neq 1 \quad (1)$$

subject to the constraints of physical capital and human capital (see details of parameters:

$$\begin{aligned} \dot{k}(t) &= \gamma k^\beta u^{1-\beta} h^{1-\beta} - \pi k - c, \quad k_0 = k(0) \\ \dot{h}(t) &= \delta(1-u)h, \quad h_0 = h(0). \end{aligned} \quad (2)$$

where  $1/\sigma$  is the constant elasticity of intertemporal substitution,  $\rho > 0$  is the discount factor,  $\beta$  is the elasticity of output with respect to physical capital,  $\gamma > 0$  is the technological levels in the goods sector,  $\delta > 0$  is the technological levels in the education sector,  $k$  is the level of physical capital,  $h$  is the level of human capital,  $c$  is per capita consumption and  $u$  is the fraction of labor allocated to the production of physical capital.

The current value Hamiltonian function is defined as

$$H(t, c, k, \lambda) = \frac{c^{1-\sigma} - 1}{1 - \sigma} + \lambda[\gamma k^\beta u^{1-\beta} h^{1-\beta} - \pi k - c] + \mu \delta(1 - u)h, \quad (3)$$

where  $\lambda(t)$  and  $\mu(t)$  are costate variables. The transversality conditions are

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) k(t) = 0, \quad \lim_{t \rightarrow \infty} e^{-\rho t} \mu(t) h(t) = 0. \quad (4)$$

The Pontrygin's maximum principle provides following set of first order conditions:

$$\lambda = c^{-\sigma}, \quad (5)$$

$$u^\beta = \frac{\gamma(1 - \beta)k^\beta h^{-\beta}}{\delta} \frac{\lambda}{\mu}, \quad (6)$$

$$\dot{k}(t) = \gamma k^\beta u^{1-\beta} h^{1-\beta} - \pi k - c, \quad (7)$$

$$\dot{h}(t) = \delta(1 - u)h, \quad (8)$$

$$\dot{\lambda} = -\lambda \gamma \beta u^{1-\beta} k^{\beta-1} h^{1-\beta} + \lambda(\rho + \pi), \quad (9)$$

$$\dot{\mu} = \mu(\rho - \delta). \quad (10)$$

The growth rates of consumption  $c$  and physical capital  $u$  are given by

$$\frac{\dot{c}}{c} = \frac{\beta \gamma}{\sigma} u^{1-\beta} k^{\beta-1} h^{1-\beta} - \frac{\rho + \pi}{\sigma}, \quad (11)$$

$$\frac{\dot{u}}{u} = \frac{(\delta + \pi)(1 - \beta)}{\beta} - \frac{c}{k} + \delta u. \quad (12)$$

The first integrals with no restriction on parameters of economy derived via partial Hamiltonian approach by Naz et al [2] are given by

$$\begin{aligned} I_1 &= \frac{\gamma(1 - \beta)}{\delta} c^{-\sigma} k^\beta u^{-\beta} h^{-\beta} e^{-(\rho - \delta)t}, \\ I_2 &= \frac{c^{-\sigma} e^{-\rho t}}{1 - \sigma} \left[ (\rho + \pi - \pi \sigma)k - \sigma c - \beta \gamma (1 - \sigma) \left( \frac{uh}{k} \right)^{1-\beta} k \right. \\ &\quad \left. + \frac{(1 - \beta)\gamma}{\delta} (\rho - \delta + \delta \sigma) \left( \frac{k}{uh} \right)^\beta h \right]. \end{aligned} \quad (13)$$

Another first integral exists under parameter restriction  $\sigma = \frac{\beta(\rho+\pi)}{2\pi\beta-\delta+\delta\beta-\pi}$  provided  $2\pi\beta - \delta + \delta\beta - \pi > 0$  to ensure that  $\sigma > 0$ . The complete analysis in terms of closed-form solution under parameter restriction  $\sigma = \frac{\beta(\rho+\pi)}{2\pi\beta-\delta+\delta\beta-\pi}$  was provided by Naz et al [2].

### 3 Closed-form solution for the Lucas-Uzawa model under fairly general values of parameters via $I_1$ and $I_2$

The closed-form solution via  $I_1$  and  $I_2$  for the case  $z \neq z^*$  case was provided in Naz et al [2] and we provide this solution in simplified form. Here, we provide closed form solutions for the original variables  $c(t), u(t), k(t), h(t), \lambda(t)$  and  $\mu(t)$  for the case  $z \neq z^*$  explicitly in terms of variable  $z(t)$ . Moreover, we provide closed form solutions for the original variables  $c(t), u(t), k(t), h(t), \lambda(t)$  and  $\mu(t)$  for the case  $z = z^*$  as well.

By setting  $I_1 = c_1$ , we obtain

$$\frac{\gamma(1-\beta)}{\delta} c^{-\sigma} k^\beta u^{-\beta} h^{-\beta} e^{-(\rho-\delta)t} = c_1, \quad (14)$$

where  $c_1$  is an arbitrary constant. Introducing  $z = \frac{hu}{k}$ , Equation (14) can be re-written as

$$z = \left( \frac{(1-\beta)\gamma}{c_1\delta} \right)^{\frac{1}{\beta}} \lambda^{\frac{1}{\beta}} e^{-\frac{(\rho-\delta)}{\beta}t}, \quad (15)$$

where  $\lambda = c^{-\sigma}$ . Equation (9) with the aid of Equation (15) yields following Bernoulli's differential equation for  $\lambda$

$$\dot{\lambda} - \lambda(\rho + \pi) = -\beta\gamma \left[ \frac{(1-\beta)\gamma}{c_1\delta} \right]^{\frac{1-\beta}{\beta}} \lambda^{\frac{1}{\beta}} e^{-\frac{(\rho-\delta)(1-\beta)}{\beta}t}, \quad (16)$$

and thus we have

$$\lambda = c^{-\sigma} = \left[ \frac{\beta\gamma^{\frac{1}{\beta}}}{\delta + \pi} \left( \frac{1-\beta}{c_1\delta} \right)^{\frac{1-\beta}{\beta}} e^{-\frac{(\rho-\delta)(1-\beta)}{\beta}t} + c_2 e^{-\frac{(1-\beta)(\rho+\pi)}{\beta}t} \right]^{\frac{\beta}{\beta-1}}, \quad (17)$$

where  $c_2$  is an arbitrary constant. We found explicit solution for  $z$  after substituting value of  $\lambda$  from Equation (17) into Equation (15)

$$z = \left[ \frac{\beta\gamma}{\delta + \pi} + c_2 \left( \frac{(1-\beta)\gamma}{c_1\delta} \right)^{\frac{\beta-1}{\beta}} e^{-\frac{(1-\beta)(\delta+\pi)}{\beta}t} \right]^{\frac{1}{\beta-1}}, \quad (18)$$

and  $z^* = \left(\frac{\beta\gamma}{\delta+\pi}\right)^{\frac{1}{\beta-1}}$  is the steady state solution. It is worthy to mention here that the system of differential equations (7)-(10) provides two sets of solutions depending on  $c_2 = 0$  and  $c_2 \neq 0$ . Thus we discuss two scenarios  
Scenario I:  $c_2 = 0$  and thus  $z = z^*$   
Scenario II:  $c_2 \neq 0$  and thus  $z \neq z^*$ .

### 3.1 Scenario I: $c_2 = 0$ and thus $z = z^*$

For  $c_2 = 0$ , we have

$$z(t) = \left(\frac{\delta + \pi}{\beta\gamma}\right)^{\frac{1}{1-\beta}} = z^*. \quad (19)$$

Equation (15) yields

$$\lambda(t) = \frac{c_1\delta}{(1-\beta)\gamma} e^{(\rho-\delta)t} z^{*\beta}, \quad (20)$$

and thus

$$c(t) = \left(\frac{(1-\beta)\gamma z^{*\beta}}{c_1\delta}\right)^{\frac{1}{\sigma}} e^{-\frac{(\rho-\delta)}{\sigma}t}. \quad (21)$$

Using initial condition  $c(0) = c_0$  and  $k(0) = k_0$ , Equations (21) gives

$$c_0 = \left(\frac{(1-\beta)\gamma}{c_1\delta z^{*\beta}}\right)^{\frac{1}{\sigma}}.$$

Equation (7) provides following solution for  $k(t)$ :

$$k(t) = \frac{\sigma\beta c_0}{\sigma\pi + \delta(\sigma - \beta) - (\pi\sigma - \rho)\beta} e^{-\frac{(\rho-\delta)}{\sigma}t} + a_1 e^{-\frac{(\pi\beta - \pi - \delta)}{\beta}t}. \quad (22)$$

The transversality condition (4) for  $k$  is satisfied provided  $\delta < \rho + \delta\sigma$  and  $a_1 = 0$ . Using initial condition  $k(0) = k_0$ , we have

$$\frac{c_0}{k_0} = \frac{\delta + \pi(1-\beta)}{\beta} - \frac{\delta - \rho}{\sigma} > 0, \quad (23)$$

as  $\frac{\delta + \pi - \pi\beta}{\beta} - \frac{\delta - \rho}{\sigma} > 0$  (see proof in Proposition 1 [1]). Equations (6) and (14) provide following expression of costate variable  $\mu$

$$\mu = c_1 e^{(\rho-\delta)t}. \quad (24)$$

Next, we set  $I_2 = c_3$  and after some simplifications, we arrive at following expression for  $h$

$$h(t) = \frac{\delta}{\gamma(1-\beta)(\rho-\delta+\delta\sigma)} z^{*\beta} \left[ c_3(1-\sigma)c_0^\sigma e^{-\delta t} + \left( \beta\gamma(1-\sigma)z^{*1-\beta}k_0 - (\rho+\pi-\pi\sigma)k_0 + \sigma c_0 \right) e^{-\frac{(\rho-\delta)}{\sigma}t} \right] \quad (25)$$

The transversality condition (4) for  $h$  is satisfied provided  $c_3 = 0$  and  $\delta < \rho + \delta\sigma$ . The initial condition  $h(0) = h_0$  yields

$$h(t) = h_0 e^{-\frac{(\rho-\delta)}{\sigma}t} \quad (26)$$

where

$$h_0 = \frac{z^*k_0\delta\sigma}{\rho-\delta+\delta\sigma}. \quad (27)$$

Finally,  $u = \frac{zk}{h}$  gives

$$u = \frac{\rho-\delta+\delta\sigma}{\delta\sigma}, \quad (28)$$

and this completes the solution. We can summarize these solutions for all variables in the following simple forms:

$$\begin{aligned} c(t) &= c_0 e^{-\frac{(\rho-\delta)}{\sigma}t}, \\ k(t) &= k_0 e^{-\frac{(\rho-\delta)}{\sigma}t}, \\ u(t) &= \frac{\rho-\delta+\delta\sigma}{\delta\sigma} = u^*, \\ h(t) &= h_0 e^{-\frac{(\rho-\delta)}{\sigma}t}, \\ \lambda(t) &= c_0^{-\sigma} e^{(\rho-\delta)t}, \\ \mu &= c_1 e^{(\rho-\delta)t}, \\ z^* &= \left( \frac{\delta+\pi}{\beta\gamma} \right)^{\frac{1}{1-\beta}}, \end{aligned} \quad (29)$$

provided  $\delta < \rho + \delta\sigma$ ,  $c_0 = \left( \frac{(1-\beta)\gamma}{c_1\delta z^{*\beta}} \right)^{\frac{1}{\sigma}}$ ,  $\frac{c_0}{k_0} = \frac{\delta+\pi(1-\beta)}{\beta} - \frac{\delta-\rho}{\sigma} > 0$ ,  $h_0 = \frac{z^*k_0}{u^*}$ .

### 3.2 Scenario II: $c_2 \neq 0$ and thus $z \neq z^*$

The expression for  $\lambda$  given in Equation (17) can be alternatively given as

$$\lambda = \frac{c_1 \delta}{(1 - \beta) \gamma} e^{(\rho - \delta)t} z^\beta. \quad (30)$$

Equation (18) for  $z(t)$ , with initial condition  $z(0) = z_0$  takes following form:

$$z(t) = \frac{z^* z_0}{[(z^{*1-\beta} - z_0^{1-\beta}) e^{-\frac{(1-\beta)(\delta+\pi)}{\beta} t} + z_0^{1-\beta}]^{\frac{1}{1-\beta}}}, \quad (31)$$

where  $c_2 = \frac{z_0^{1-\beta} - z^{*1-\beta}}{(\frac{(1-\beta)\gamma}{c_1 \delta})^{\frac{\beta-1}{\beta}}}$ .

The variable  $c(t)$  with  $c(0) = c_0$  takes following form:

$$c = c_0 z_0^{\frac{\beta}{\sigma}} e^{-\frac{(\rho-\delta)}{\sigma} t} z^{-\frac{\beta}{\sigma}}, \quad c_0 z_0^{\frac{\beta}{\sigma}} = \left( \frac{c_1 \delta}{(1 - \beta) \gamma} \right)^{-\frac{1}{\sigma}} \quad (32)$$

where  $z$  is same as given in (31). The differential equation (7) for  $k$  results in following integrable differential equation

$$\dot{k} + (\pi - \gamma z^{1-\beta}) k = -c_0 z_0^{\frac{\beta}{\sigma}} e^{-\frac{(\rho-\delta)}{\sigma} t} z^{-\frac{\beta}{\sigma}}, \quad (33)$$

and it provides

$$k(t) = \left( a_3 - \frac{c_0 z_0^{\frac{\beta}{\sigma}}}{(\pi + \delta)^{\frac{1}{1-\beta}}} F(t) \right) (\pi + \delta)^{\frac{1}{1-\beta}} z(t)^{-1} e^{\frac{(\pi + \delta - \pi \beta)}{\beta} t}, \quad (34)$$

where

$$F(t) = \int_0^t z(t)^{\frac{\sigma-\beta}{\sigma}} e^{-(\frac{\delta+\pi-\pi\beta}{\beta} - \frac{\delta-\rho}{\sigma})t} dt \quad (35)$$

$a_3$  is arbitrary constant of integration and  $\frac{\delta+\pi-\pi\beta}{\beta} - \frac{\delta-\rho}{\sigma} > 0$  (see proof in Proposition 1 [1]). The initial condition  $k(0) = k_0$  yields  $a_3 = \frac{k_0 z_0}{(\pi + \delta)^{\frac{1}{1-\beta}}}$  and thus expression for  $k(t)$  simplifies to the following form:

$$k(t) = \left( \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - F(t) \right) c_0 z_0^{\frac{\beta}{\sigma}} z(t)^{-1} e^{\frac{(\delta+\pi-\pi\beta)}{\beta} t}. \quad (36)$$



The transversality condition (4) for  $k$  is satisfied provided  $\delta < \rho + \delta\sigma$  and

$$\lim_{t \rightarrow \infty} F(t) = \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}}. \quad (37)$$

It is important to mention here that the integrand of  $F(t)$  is positive and bounded therefore  $\lim_{t \rightarrow \infty} F(t)$  is a finite number. Equation (6) provides following expression for the costate variable

$$\mu = c_1 e^{(\rho-\delta)t}. \quad (38)$$

Setting  $I_2 = c_3$ , we find

$$\begin{aligned} \frac{c^{-\sigma} e^{-\rho t}}{1-\sigma} \left[ (\rho + \pi - \pi\sigma)k - \sigma c - \beta\gamma(1-\sigma)\left(\frac{uh}{k}\right)^{1-\beta}k \right. \\ \left. + \frac{(1-\beta)\gamma}{\delta}(\rho - \delta + \delta\sigma)\left(\frac{k}{uh}\right)^\beta h \right] = c_3 \end{aligned} \quad (39)$$

and this gives

$$h(t) = \frac{\left( \frac{c_3(1-\sigma)}{\lambda e^{-\rho t}} - (\rho + \pi - \pi\sigma)k + \sigma c + \beta\gamma(1-\sigma)z^{1-\beta}k \right) \delta z^\beta}{\gamma(1-\beta)(\rho - \delta + \delta\sigma)}. \quad (40)$$

The transversality condition for  $h$  requires to choose  $c_3 = 0$  and  $\delta < \rho + \delta\sigma$ . Finally, the variable  $u$  can be determined from  $u = zk/h$  and it simplifies to

$$u(t) = \frac{\delta^{-1}\gamma(1-\beta)(\rho - \delta + \delta\sigma)\left(\frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - F(t)\right)}{[\beta\gamma(1-\sigma) - (\rho + \pi - \pi\sigma)z^{\beta-1}]\left(\frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - F(t)\right) + \sigma z^{\beta-\frac{\beta}{\sigma}} e^{-\left(\frac{\delta+\pi-\pi\beta}{\beta} - \frac{\delta-\rho}{\sigma}\right)t}}.$$

This completes the solution. We then apply the initial conditions  $h(0) = h_0$ ,

$u_0 = \frac{z_0 k_0}{h_0}$  and we summarize the solutions for all variables as follows:

$$\begin{aligned}
c(t) &= c_0 z_0^{\frac{\beta}{\sigma}} e^{-\frac{(\rho-\delta)}{\sigma} t} z^{-\frac{\beta}{\sigma}}, \\
k(t) &= \left( \frac{k_0}{c_0 z_0^{\frac{\beta}{\sigma}}} - F(t) \right) c_0 z_0^{\frac{\beta}{\sigma}} z(t)^{-1} e^{\frac{(\delta+\pi-\pi\beta)}{\beta} t}, \\
h(t) &= \frac{h_0}{z_0 [\sigma c_0 z_0^{\beta-1} - (\rho + \pi - \pi\sigma) k_0 z_0^{\beta-1} + \beta\gamma(1-\sigma)k_0]} \left[ \sigma c_0 z_0^{\frac{\beta}{\sigma}} e^{-\frac{(\rho-\delta)}{\sigma} t} z^{-\frac{\beta}{\sigma} + \beta} \right. \\
&\quad \left. + (\beta\gamma(1-\sigma) - (\rho + \pi - \pi\sigma) z^{\beta-1}) \left( \frac{k_0}{c_0 z_0^{\frac{\beta}{\sigma}}} - F(t) \right) c_0 z_0^{\frac{\beta}{\sigma}} e^{\frac{(\delta+\pi-\pi\beta)}{\beta} t} \right], \\
u(t) &= \frac{u_0}{k_0} [\sigma c_0 z_0^{\beta-1} - (\rho + \pi - \pi\sigma) k_0 z_0^{\beta-1} + \beta\gamma(1-\sigma)k_0] \\
&\quad \left( \frac{k_0}{c_0 z_0^{\frac{\beta}{\sigma}}} - F(t) \right) \\
&\times \frac{1}{[\beta\gamma(1-\sigma) - (\rho + \pi - \pi\sigma) z^{\beta-1}] \left( \frac{k_0}{c_0 z_0^{\frac{\beta}{\sigma}}} - F(t) \right) + \sigma z^{\beta-\frac{\beta}{\sigma}} e^{-(\frac{\delta+\pi-\pi\beta}{\beta} - \frac{\delta-\rho}{\sigma}) t}}, \\
\lambda(t) &= \frac{c_1 \delta}{(1-\beta)\gamma} e^{(\rho-\delta)t} z^{\beta}, \\
\mu(t) &= c_1 e^{(\rho-\delta)t},
\end{aligned}$$

where

$$\begin{aligned}
F(t) &= \int_0^t z(t)^{\frac{\sigma-\beta}{\sigma}} e^{-(\frac{\delta+\pi-\pi\beta}{\beta}-\frac{\delta-\rho}{\sigma})t} dt, \\
z(t) &= \frac{z^* z_0}{[(z^{*1-\beta} - z_0^{1-\beta})e^{-\frac{(1-\beta)(\delta+\pi)}{\beta}t} + z_0^{1-\beta}]^{\frac{1}{1-\beta}}}, \\
\lim_{t \rightarrow \infty} F(t) &= \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}}, \\
\rho < \delta < \rho + \delta\sigma, \quad \frac{\delta + \pi - \pi\beta}{\beta} - \frac{\delta - \rho}{\sigma} &> 0, \\
c_0 z_0^{\frac{\beta}{\sigma}} &= \left( \frac{c_1 \delta}{(1-\beta)\gamma} \right)^{-\frac{1}{\sigma}}, \\
\frac{\gamma(1-\beta)(\rho - \delta + \delta\sigma)}{\delta} &= \frac{u_0}{k_0} [\sigma c_0 z_0^{\beta-1} - (\rho + \pi - \pi\sigma)k_0 z_0^{\beta-1} + \beta\gamma(1-\sigma)k_0], \\
z^* &= \left( \frac{\beta\gamma}{\delta + \pi} \right)^{\frac{1}{\beta-1}}.
\end{aligned} \tag{41}$$

The first integrals  $I_1$  and  $I_2$  yield two solutions of the dynamical system of ODEs (7)-(12) given in equations (29) and (41) for fairly general values of  $\sigma$  and  $\beta$ . It is straight forward to show that

$$\lim_{t \rightarrow \infty} u(t) = u^*. \tag{42}$$

## 4 Closed-form solution for the Lucas-Uzawa model under fairly general values of parameters via $I_1$

Now we show how one can utilize only one first integral  $I_1$  to derive the same closed-form solution as in the existing literature which was also derived by Chilarescu [1] via the classical approach. By setting  $I_1 = a_1$ , we will arrive at equations (14)-(18). Here also the following two scenarios will arise:

Case I:  $a_2 = 0$   
Case II:  $a_2 \neq 0$ .

Equations (19)-(24) for variables  $c(t)$ ,  $k(t)$ ,  $\lambda(t)$  and  $\mu(t)$  will follow from the previous section. Now instead of utilizing the second first integral, we proceed as follows to derive closed form solutions for  $h(t)$  and  $u(t)$ .

Equation (12) simplifies to

$$\frac{\dot{u}}{u} = \frac{\delta - \rho - \delta\sigma}{\sigma} + \delta u. \quad (43)$$

Equation (43) gives

$$u(t) = \frac{\frac{\delta - \rho - \delta\sigma}{\sigma}}{a_2 \left( \frac{\delta - \rho - \delta\sigma}{\sigma} \right) e^{-\frac{\delta - \rho - \delta\sigma}{\sigma} t} - \delta}. \quad (44)$$

Finally,  $h = \frac{zk}{u}$  gives

$$h(t) = \frac{a_2 \left( \frac{\delta - \rho - \delta\sigma}{\sigma} \right) e^{-\frac{\delta - \rho - \delta\sigma}{\sigma} t} - \delta}{\frac{\delta - \rho - \delta\sigma}{\sigma}} z^* k_0 e^{-\frac{(\rho - \delta)}{\sigma} t}. \quad (45)$$

The transversality condition (4) for  $h$  is satisfied provided  $\delta < \rho + \delta\sigma$  and  $a_2 = 0$ . Thus we arrive at the same solution for all variables as given by Equation (29).

For  $a_2 \neq 0$ , we will follow same procedure as described in previous section to derive equations (30)-(38). Substituting  $c(t)$  and  $k(t)$  from equations (32) and (36) into equation (12), we have

$$\frac{\dot{u}}{u} = \frac{(\delta + \pi)(1 - \beta)}{\beta} - \frac{F'(t)}{\frac{k_0}{c_0 z_0^{\frac{\beta - \sigma}{\sigma}}} - F(t)} + \delta u. \quad (46)$$

The solution of equation (46) with initial condition  $u(0) = u_0$  is given by

$$u(t) = \frac{\frac{(\delta + \pi)(1 - \beta)}{\beta} u_0 \left[ \frac{k_0}{c_0 z_0^{\frac{\beta - \sigma}{\sigma}}} - F(t) \right]}{\left[ \left( \frac{(\delta + \pi)(1 - \beta)}{\beta} + \delta u_0 \right) \frac{k_0}{c_0 z_0^{\frac{\beta - \sigma}{\sigma}}} - \delta u_0 G(t) \right] e^{-\frac{(\delta + \pi)(1 - \beta)}{\beta} t} - \delta u_0 \left[ \frac{k_0}{c_0 z_0^{\frac{\beta - \sigma}{\sigma}}} - F(t) \right]}, \quad (47)$$

where

$$G(t) = \int_0^t z(t)^{\frac{\sigma-\beta}{\sigma}} e^{-\frac{\delta\sigma-\delta-\rho}{\sigma}t} dt. \quad (48)$$

The solution (47) holds provided

$$\lim_{t \rightarrow \infty} \left[ \left( \frac{(\delta + \pi)(1 - \beta)}{\beta} + \delta u_0 \right) \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - \delta u_0 G(t) \right] = 0, \quad (49)$$

with

$$\lim_{t \rightarrow \infty} G(t) = \frac{\left( \frac{(\delta + \pi)(1 - \beta)}{\beta} + \delta u_0 \right)}{\delta u_0} \lim_{t \rightarrow \infty} F(t), \quad (50)$$

and  $\lim_{t \rightarrow \infty} F(t)$  is given in (37). The variable  $h$  can be determined from  $h = zk/u$  and is given by

$$\begin{aligned} h(t) = & \left[ \left( \frac{(\delta + \pi)(1 - \beta)}{\beta} + \delta u_0 \right) \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - \delta u_0 G(t) \right] e^{-\frac{(\delta + \pi)(1 - \beta)}{\beta}t} \\ & - \delta u_0 \left[ \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - F(t) \right] \times \frac{c_0 z_0^{\frac{\beta}{\sigma}}}{\frac{(\delta + \pi)(1 - \beta)}{\beta} u_0} e^{\frac{(\pi + \delta - \pi\beta)}{\beta}t}, \end{aligned} \quad (51)$$

We can summarize the solutions for all variables as follows:

$$\begin{aligned} c(t) &= c_0 z_0^{\frac{\beta}{\sigma}} e^{-\frac{(\rho - \delta)}{\sigma}t} z^{-\frac{\beta}{\sigma}}, \\ k(t) &= \left( \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - F(t) \right) c_0 z_0^{\frac{\beta}{\sigma}} z(t)^{-1} e^{\frac{(\pi + \delta - \pi\beta)}{\beta}t}, \\ h(t) &= \left[ \left( \frac{(\delta + \pi)(1 - \beta)}{\beta} + \delta u_0 \right) \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - \delta u_0 G(t) \right] e^{-\frac{(\delta + \pi)(1 - \beta)}{\beta}t} \\ &\quad - \delta u_0 \left[ \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - F(t) \right] \times \frac{c_0 z_0^{\frac{\beta}{\sigma}}}{\frac{(\delta + \pi)(1 - \beta)}{\beta} u_0} e^{\frac{(\pi + \delta - \pi\beta)}{\beta}t}, \\ u(t) &= \frac{\frac{(\delta + \pi)(1 - \beta)}{\beta} u_0 \left[ \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - F(t) \right]}{\left[ \left( \frac{(\delta + \pi)(1 - \beta)}{\beta} + \delta u_0 \right) \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - \delta u_0 G(t) \right] e^{-\frac{(\delta + \pi)(1 - \beta)}{\beta}t} - \delta u_0 \left[ \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - F(t) \right]} \\ \lambda(t) &= \frac{c_1 \delta}{(1 - \beta) \gamma} e^{(\rho - \delta)t} z^\beta, \\ \mu(t) &= c_1 e^{(\rho - \delta)t}, \end{aligned}$$

where

$$\begin{aligned}
\rho < \delta < \rho + \delta\sigma, \frac{\delta + \pi - \pi\beta}{\beta} - \frac{\delta - \rho}{\sigma} > 0, \\
F(t) &= \int_0^t z(t)^{\frac{\sigma-\beta}{\sigma}} e^{-(\frac{\delta+\pi-\pi\beta}{\beta}-\frac{\delta-\rho}{\sigma})t} dt, \\
G(t) &= \int_0^t z(t)^{\frac{\sigma-\beta}{\sigma}} e^{-\frac{\delta\sigma-\delta+\rho}{\sigma}t} dt, \\
z(t) &= \frac{z^* z_0}{[(z^{*1-\beta} - z_0^{1-\beta})e^{-\frac{(1-\beta)(\delta+\pi)}{\beta}t} + z_0^{1-\beta}]^{\frac{1}{1-\beta}}}, \\
c_0 z_0^{\frac{\beta}{\sigma}} &= \left( \frac{c_1 \delta}{(1-\beta)\gamma} \right)^{-\frac{1}{\sigma}}, \\
\lim_{t \rightarrow \infty} F(t) &= \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}}, \\
\lim_{t \rightarrow \infty} \left[ \left( \frac{(\delta+\pi)(1-\beta)}{\beta} + \delta u_0 \right) \frac{k_0}{c_0 z_0^{\frac{\beta-\sigma}{\sigma}}} - \delta u_0 G(t) \right] &= 0, \\
\lim_{t \rightarrow \infty} G(t) &= \frac{(\frac{(\delta+\pi)(1-\beta)}{\beta} + \delta u_0)}{\delta u_0} \lim_{t \rightarrow \infty} F(t), \\
z^* &= \left( \frac{\beta\gamma}{\delta+\pi} \right)^{\frac{1}{\beta-1}}.
\end{aligned} \tag{52}$$

The closed form solution (52) derived by only utilizing  $I_1$  is exactly the same as derived by Chilarescu [1] via the classical approach.

## 5 Comparison of Closed-form Solutions for the Lucas-Uzawa model

The closed form solutions of the Lucas-Uzawa model have been derived in the literature by using both the newly developed partial Hamiltonian approach and the classical approach. The partial Hamiltonian approach utilizes Lie group theoretical techniques to construct a closed-form solution. Using this partial Hamiltonian methodology, we have established three sets of closed

form solutions (29), (41) and (52) for the Lucas-Uzawa model with no parameter restrictions. For the  $z = z^*$  case only one solution arises which is given in (29) whereas for the  $z \neq z^*$  case we obtained two solutions which are given in (41) and (52).

We have shown how first integrals derived via the partial Hamiltonian approach can be utilized to construct multiple closed form solutions. It is shown that for the case where  $z = z^*$ , the closed form solution (29) which was previously derived in the literature using the classical approach (see Chilarescu [1]) can also be constructed by utilizing either one first integral  $I_1$  or by using two first integrals  $I_1$  and  $I_2$ . For the case where  $z \neq z^*$ , we have obtained the second solution (52) found by Chilarescu [1] and this solution was obtained by utilizing only one first integral  $I_1$ . Using our partial Hamiltonian methodology, we also arrive at an additional solution (41) in the case where  $z \neq z^*$  by utilizing the two first integrals  $I_1$  and  $I_2$ .

Thus there exists three sets of closed form solutions (29), (41) and (52) for the Lucas-Uzawa model with no parameter restrictions. Chilarescu [1] utilized the classical approach to provide two solutions with no restrictions on parameters which were the same as the solutions in (29) and (52) obtained using the partial Hamiltonian approach. The partial Hamiltonian approach also provided one additional solution given in (41) which is new to the literature.

In the long run all these solutions converge to the same steady state which is important in the context of economic growth theory and gives rise to multiple equilibria. For the  $z = z^*$  case, it is straightforward to compute this simple closed form solution by any of the existing techniques in the literature. At the same time, the case in which  $z \neq z^*$  is also important. A comparison of the closed-form solutions (41) and (52) in the second case shows that the expressions for consumption  $c$ , physical capital stock  $k$ , and the costate variables  $\lambda$  and  $\mu$  are the same in both solutions. On the other hand, the expressions for the fraction of labor devoted to physical capital,  $u$ , and the level of human capital,  $h$ , are different in our newly obtained solution. Another important result is that the previously obtained closed form solution (52) involves two numerically computable functions  $F(t)$  and  $G(t)$  whereas our newly closed form solution (41) involves only one numerically computable function  $F(t)$ . So our newly derived closed form solution (41) which was obtained from the partial Hamiltonian approach is fundamentally different and is also in a simpler form than the previously obtained solution (52).

It is straight forward to show, for the closed form solutions (29), (41) and

(52), that

$$\lim_{t \rightarrow \infty} u(t) = u^*. \quad (53)$$

It is worthy to mention here that l'Hôpital rule is applied to establish  $\lim_{t \rightarrow \infty} u(t)$ . The growth rates of consumption,  $c$ , physical capital,  $k$ , and human capital,  $h$ , decrease over time and approach  $\frac{\delta - \rho}{\sigma}$  as  $t \mapsto \infty$ . Also, the growth rate of the fraction of labor allocated to the production of physical capital,  $u$ , approaches zero as  $t \mapsto \infty$  whereas the growth rates of both costate variables,  $\lambda$  and  $\mu$ , equal  $(\rho - \delta)$  as  $t \mapsto \infty$ .

**Remark 1:** It is important to mention here that partial Hamiltonian approach provided a new specific restriction  $\sigma = \frac{\beta(\rho + \pi)}{2\pi\beta - \delta + \delta\beta - \pi}$  on parameters for which another first integral was established. Naz et al [2] provided detailed analysis of closed-form solution under this specific restriction. The closed form solution under this specific restriction was not derived before in literature.

**Remark 2:** It is worthy to mention here that one cannot derived closed-form solutions for the case  $\sigma = \beta$  directly from (29), (41) and (52). We need to derive solutions by utilizing  $I_1$  and  $I_2$  or only  $I_1$ .

## 6 Conclusions

In this paper we establish closed-form solutions for the Lucas-Uzawa model with the aid of the partial Hamiltonian approach and then compare our results with those derived by the classical approach. The partial Hamiltonian approach provides two first integrals [2] in the case where there are no parameter restrictions. We utilize these two first integrals to construct closed form solutions for all the variables in the model for two different scenarios: (i)  $z = z^*$  and (ii)  $z \neq z^*$  where  $z(t) = \frac{h(t)u(t)}{k(t)}$ .

For the  $z = z^*$  case, both the partial Hamiltonian approach and the classical approach provide one solution. For the  $z \neq z^*$  case, the classical approach yields one solution while the the partial Hamiltonian approach provides the same solution as well as one additional solution which is completely new to the literature. In the newly obtained solution, the expressions for the levels of consumption and capital stock are equal to those found in the older solution, while the amount of labor allocated to the production of physical capital and the level of human capital are different from those values found in the older solution. The existence of three closed form solutions is new to



the literature. We also show that these equilibria all converge to the same long run balanced growth path.

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